Analytic solution of the nonlinear equation

Hyuk-jae Lee, Seok Lee, Deok Ha Woo, Taik Jin Lee, and Jae Hun Kim*

Korea Institute of Science and Technology
Seoul, 130-650, Korea, jaekim@kist.re.kr

***Abstract*- We analytically solve the nonlinear wave equation of the beam, which travels through the nonlinear Kerr medium. The tanh function method, a powerful method solving the traveling wave equation, is applied to the self-guiding light.**

I. INTRODUCTION

The electromagnetic fields are satisfied with the Maxwell's and wave equations. In the case that light travels through a nonlinear medium, we should consider nonlinear effect from the nonlinear medium. However, it is very hard to analytically solve the nonlinear equation. Numerical approach has been developed to solve the nonlinear traveling electromagnetic field [1]. The split-step Fourier method is well known as the approach understanding the nonlinear effects in the optical fibers [2]. However, this method takes a long time for obtaining the results. Thus, several different finite-difference schemes were suggested as other methods to analyze nonlinear effects [3]. The inverse scattering method was applied to nonlinear partial differential equation for analytic solution [4]. However, the applications of this method are limited to some specific cases.

Recently, the tanh function method, applicable to various cases, is introduced as a new and most effective method for finding exact solutions of a nonlinear wave equation satisfied with a traveling wave through the nonlinear medium [5]. Furthermore, some studies showed that the tanh function method could be extended to the spatial types of nonlinear equations [6]. The reason that these methods were possible is due to powerful software that can deal with the tedious algebraic computation.

In this paper, we applied the generalized tanh function method to find the analytic solution of the nonlinear equation describing the traveling wave through the nonlinear Kerr medium.

II. SOLVING THE NONLINEAR PARTIAL DEFFERENTIAL EQUATION

If we consider the response of the optical beam traveling through the substantial thickness of nonlinear homogeneous medium to a harmonic electric field, the self-guiding of light is described mathematically by Helmholtz equation. For simplification [7], we assume that the electric field varies slowly in the z-direction and does not vary in the y-direction. Using the approximation and the inverse Fourier transformation in time, the Helmholtz equation becomes

$$\frac{\partial^2 E}{\partial x^2} - i\alpha\frac{\partial E}{\partial z} + \beta\frac{\partial^2 E}{\partial t\partial z} + \chi(|E|^2 E) = 0. \qquad (1)$$

For the simplification, we used the constants α, β and χ for complex constants.

For applying to the tanh function method, the first step is to introduce the wave transformation $E(x,z,t) = U(\xi)$, $\xi = x + \eta z + \lambda t$, and changes (1) to an ordinary differential equation. The next step is to introduce a new variable $T = T(\xi)$, which is the solution of the Riccati equation, $T' = k + T^2$. Hence all derivatives of T with respective to ξ are polynomial in T as

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$T'' = -2T(k + T^2)$.

To find the solution, we propose the polynomial solution $U(T) = \sum_{v=0}^{M} a_v T^v$, where the positive integer M can be determined by balancing the highest derivative term with nonlinear terms. Here we can obtain $M = 1$ after simple calculation. We try the polynomial solution as

$$U(T) = a_0 + a_1 T. \qquad (2)$$

After we substitute (2) into the ordinary differential equation of (1), the constants $k, \eta, \lambda, a_0, a_1$ are obtained as setting all coefficients of T^v to zero. Then we get the algebraic equations of the constants. a_0, a_1 and k are given by solving the algebraic equation,

$$a_0 = 0, \; a_1 = -\frac{\sqrt{2(1+\eta\lambda\beta)}}{\sqrt{\chi}}, \; k = \frac{(\alpha\eta - b\eta\beta)^2}{8(1+\eta\lambda\beta)} \qquad (3)$$

and the others are arbitrary.

The final solution of (1) is given by

$$E(x,z,t) = \frac{\sqrt{2}\sqrt{k(1+\eta\lambda\beta)}}{\sqrt{\chi}} e^{i(ax+k_0 z+bt)} \times \tanh(\sqrt{k}(x+\eta z+\lambda t)), \qquad (4)$$

where $a = \frac{1}{2}(\alpha\eta - b\eta\beta)$ and $b, c. \eta, \lambda$ are arbitrary constants. Figure 1 shows the wave propagation based on (4).

III. Conclusion

We showed that the generalized tanh function method can be applied to the nonlinear partial differential equation describing the traveling light through nonlinear homogeneous medium. This shows that the generalized tanh function method is more applicable than the original tanh function method. Equation (2) describes the traveling wave that is similar to soliton solution that travels steadily without a variation.

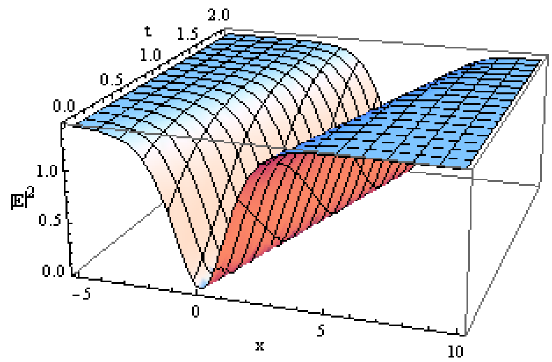

Fig. 1. Propagation of the square of (4) after fixing the z coordinate and the constants

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References

[1] Govind and P. Agrawal, *Nonlinear Fiber Optics,* 3rd ed.,Academic Press, San Diego, CA, 2001.

[2] R. A. Fisher and W. K. Bischel, "The role of linear dispersion in plane-wave self-phase modulation," App. Phys, Lett. Vol. 23, pp661-663, December 1973.

[3] T. R Taha and M. I. Ablowitz, "Analytical and numerical aspects of certain nonlinear evolution equations. II. Numerical, nonlinear Schrödinger equation," J. Comput. Phys. vol. 55, pp203-230, Agust 1984.

[4] W. Malfliet, "Solitary wave solutions of nonlinear wave equations," *Amer. J. Phys*. vol. 60, pp. 650-654, July 1992.

[5] E. Parkers, "Eaxt solutions to the two-dimensional Korteweg-de Vries-Burgers equation," *J. Phys*. vol. A 27, pp. L497-L501, July 1994.

[6] S. A. El-Wakil and M. A. Abdou, "Modified extended tanh function method for solving nonlinear partial differential equation," Chaos, Solitons & Fractals, vol. 31, pp321-330, 2007.

[7] B. E. A Saleh and M. C. Teich, *Fundamentals of Photonics.* A Wiley-interscience publication, New York, pp754, 1991